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Almost periodically unitary stochastic processes

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A continuous second order complex process $\{X(t), t\in\mathbb{R}\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ is called almost periodically unitary (APU) if there exists a strongly continuous one parameter group of unitary operators $\{U(\tau), \tau\in\mathbb{R}\}$ for which the set $S(\varepsilon, X, U)=\{\tau\colon \sup_{t\in\mathbb{R}} \|X(t+\tau)-U(\tau)X(t)\|_{L_2}<\varepsilon\}$ is relatively dense (has bounded gaps) for every $\varepsilon>0$. These processes include continuous stationary processes for which $S(\varepsilon, X, U)=\mathbb{R}$, continuous periodically correlated processes for which $S(\varepsilon, X, U)\supset\{jT, j\in\mathbb{Z}\}$ for some real T, and the L_2-valued uniformly almost periodic functions for which $U(\tau)=I$. In this paper, we show that $X(t)$ is APU if and only if $X(t)=U(t)[P(t)]$ where $P(t)$ is an L_2-valued uniformly almost periodic function. Examples are given and basic properties motivated by the theory of uniformly almost periodic functions are provided. These processes are shown to be uniformly almost periodically correlated and hence almost periodically correlated in the sense of Gladyshev. We give representations for the processes based on the spectral theory for unitary groups and on the harmonic analysis of uniformly almost periodic functions. Finally, we give an analysis of the correlation functions in terms of the representation theory, and show that every APU process is a uniform limit of a sequence of strongly harmonizable processes.

1. Introduction

This paper is concerned with a class of continuous-time nonstationary second order stochastic processes that are closely related to the *almost periodically correlated* (APC) processes introduced by Gladyshev [13] and the *uniformly almost periodically correlated* (UAPC) that have been treated more recently by Dehay [8]; the latter were also discussed briefly by Isokawa [20] and were introduced earlier by Gardner [10], who called them *almost cyclostationary.* Additional facts concerning the correlation theory of APC processes may be found in Hurd [17]; particular types of APC processes, called biperiodically correlated, are examined by Dragan [9]. A particular model of an APC sequence is examined by Molajo [26].

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Although the existing papers on PC and almost PC processes treat various aspects of the representation of the processes and their covariances, none appear to have made the connection between these processes and groups of unitary operators, although a connection to multi-variate stationary processes has been known from the beginning (see Gladyshev [12, 13]). In this paper we show that every PC process has an imbedded unitary operator and use this fact to motivate the *almost periodically unitary* (APU) processes. We provide example of APU processes and then give a characterization in terms of groups of unitary operators and vector valued almost periodic functions. We determine some of their basic properties and extend to these processes some of the classical results motivated by UAP functions. We show that APU processes are UAPC and thus also APC the sense of Gladyshev. Next, we give representations based on the spectral theory for groups of unitary operators and harmonic analysis for uniformly almost periodic functions. The representations obtained show the close relationship between APU processes and continuous parameter stationary vector processes. We sketch the correlation theory resulting from the representations, and relate the results to the general correlation theory for APC processes [17]. Finally, we show that every APU process is a uniform limit of a sequence of strongly harmonizable processes.

Since APU processes are also APC, the recent results on consistent estimation of the Fourier coefficients of the correlation function and their spectral densities will apply to these processes, provided other conditions are met [18, 19, 23]. In addition, a weak law of large numbers for APC processes [5] may also apply.

In all that follows, $\{X(t), t \in \mathbb{R}\}$ refers to a complex valued process defined on a probability space $(\Omega, \mathcal{F}, P)$ with $E|X(t)|^2 < \infty$ and $E\{X(t)\} \equiv 0$ for all t. For any two times (s, t), the correlation function is given by

$$R(s, t) = E\{X(s)\overline{X(t)}\} = \langle X(s), X(t)\rangle_{L_2} \tag{1.1}$$

where $\langle \cdot, \cdot \rangle_{L_2}$ denotes the usual $L_2(\Omega)$ inner product. The Hilbert space of the process, $H(X)$ is the closure, in $L_2(\Omega)$, of finite linear combinations of the form $\sum_{j=1}^{N} \alpha_j X(t_j)$ and we write $H(X) = \overline{\text{sp}}\{X(t), t \in \mathbb{R}\}$. $H(X)$ inherits its inner product from $L_2(\Omega)$ so that for y_1, y_2 in $H(X)$, $\langle y_1, y_2\rangle_{H(X)} = \langle y_1, y_2\rangle_{L_2(\Omega)}$ and $\|y_1\|_{H(X)} = \|y_1\|_{L_2(\Omega)}$.

A second order process is said to be wide sense stationary if (and only if) $R(s, t)$ depends only on $s-t$. In this event, a family $\{U(\tau), \tau \in \mathbb{R}\}$ of operators may be defined, for every $\tau \in \mathbb{R}$, on the span of $X(t)$ by

$$U(\tau)\left[\sum_{j=1}^{N} \alpha_j X(t_j)\right] = \sum_{j=1}^{N} \alpha_j X(t_j + \tau) \tag{1.2}$$

and it follows that each $U(\tau)$ is unitary. Further, $U(\tau)$ may be extended to $H(X)$ and if $X(t)$ is continuous, the collection $\{U(\tau), \tau \in \mathbb{R}\}$ forms a strongly continuous one parameter group (sometimes called a shift group) indexed on $\mathbb{R}$. The group property arises from the fact that the composition, $U(\tau_1)U(\tau_2)$, is defined by $U(\tau_1 + \tau_2)$ for every $\tau_1, \tau_2 \in \mathbb{R}$ and clearly $U(0) = I$. The spectral representation of a continuous group of unitary operators, due to Stone [30], may be used to obtain [22, 28] the spectral representation of wide sense stationary processes. We next see how shift groups are related to PC processes.

A second order process is said to be periodically correlated with period T if

$$R(s, t) = R(s+T, t+T) \tag{1.3}$$

for every (s, t) [13]. In this event, a collection of unitary (shift) operators may be defined in the same manner, but only for $\tau = nT$, $n \in \mathbb{Z}$. That is, for any n,

$$U(n)\left[\sum_{j=1}^{N} \alpha_j X(t_j)\right] = \sum_{j=1}^{N} \alpha_j X(t_j + nT) \tag{1.4}$$

defines a unitary operator $U(n)$ that may be extended to $H(X)$, and $\{U(n), n \in \mathbb{Z}\}$ forms a one parameter shift group under composition, $U(n_1 + n_2) = U(n_1)U(n_2)$ and $U(0) = I$.

To complete the background and motivation, we now review the definition of almost periodic functions, according to Bohr [4] (see also Besicovitch [3] or Corduneanu [7]. A continuous function $f: \mathbb{R} \to \mathbb{R}$ is said to be uniformly almost periodic (UAP) if for every $\varepsilon > 0$, the set

$$S(\varepsilon, f) = \left\{\tau\colon \sup_{t \in \mathbb{R}} |f(t+\tau) - f(t)| < \varepsilon\right\} \tag{1.5}$$

is relatively dense (RD); a set $E \subset \mathbb{R}$ is RD if some $L > 0$, every interval of length L has a non-null intersection with E. Such an L shall be called an inclusion length for E and we note that a set E is RD if and only if it has bounded gaps. The notion of almost periodic functions has been extended in several ways. If f is a function with values in a Banach space B (or more specifically, a Hilbert space) with norm $\|\cdot\|_{\mathrm{B}}$, (1.5) may be replaced with (see Corduneanu [7] or Amerio and Prouse [1])

$$S(\varepsilon, f) = \left\{\tau\colon \sup_{t \in \mathbb{R}} \|f(t+\tau) - f(t)\|_{\mathrm{B}} < \varepsilon\right\}, \tag{1.6}$$

and in this case we would say that $f: \mathbb{R} \to B$ is a UAP function. Both (1.5) and (1.6) require uniform approximation; weaker notions of convergence have been introduced to treat functions that are not continuous (see Besicovitch [3] for a summary).

This work was motivated by explicitly writing the meaning of $X(t+\tau) = U(t)[X(t)]$ for stationary and PC processes, and then extending the meaning to get an elaboration of (1.6). Note for stationary processes, the translated $X(\cdot + \tau)$ are given *exactly* (up to an equivalence) by the action of a unitary operator $U(\tau)$; that is,

$$\|X(t+\tau) - U(\tau)X(t)\|_{L_2} = 0 \quad \text{for all } \tau \text{ and } t \tag{1.7}$$

where, to be more precise, $U(\tau)X(t)$ means $U(\tau)[X(t)]$. For PC processes we find similarly that

$$\|X(t+nT)-U(n)X(t)\|_{L_2}=0 \quad \text{for all } n \text{ and } t. \tag{1.8}$$

Stating again the meaning of (1.6), a function is UAP if for every $\varepsilon>0$ the translates $f(\cdot+\tau)$, for τ in a RD set $S(\varepsilon, f)$, can be uniformly approximated within ε by f. These observations suggest that we examine the second order processes for which the set

$$S(\varepsilon, X, U)=\left\{\tau\colon \sup_{t\in\mathbb{R}} \|X(t+\tau)-U(\tau)X(t)\|_{L_2}<\varepsilon\right\} \tag{1.9}$$

is RD for every $\varepsilon>0$, and where $\{U(\tau), \tau\in\mathbb{R}\}$ is a family of unitary operators $\{U(\tau)\colon L_2(\Omega)\to L_2(\Omega), \tau\in\mathbb{R}\}$. Subject to the following remarks concerning continuity, this condition defines the APU processes.

The notions of UAP functions rely heavily on continuity and so we are lead to imposing continuity on $X(t)$ (as a mapping from $\mathbb{R}$ to $L_2(\Omega)$) and some additional restriction on $\{U(\tau), \tau\in\mathbb{R}\}$. We require that $\{U(\tau), \tau\in\mathbb{R}\}$ be a strongly continuous shift group; that is, for all x (in some subset of L_2) and any $\varepsilon>0$, there is a $\delta(x,\tau)$ for which $\|[U(\tau+h)-U(\tau)]x\|<\varepsilon$ whenever $|h|<\delta(x,\tau)$. Note $U(\tau_1+\tau_2)=U(\tau_1)U(\tau_2)$ and the unitarity of $U(\tau)$ means that continuity at $\tau=0$ suffices for uniform continuity with respect to τ.

Definition. A second order process X is called *almost periodically unitary* if it is continuous (in $\|\cdot\|_{L_2}$) and if there exists a strongly continuous unitary group $\{U(\tau), \tau\in\mathbb{R}\}$ such that the set $S(\varepsilon, X, U)$, for every $\varepsilon>0$, is RD.

This notion no longer requires, as in the theory of UAP functions, the translates $X(\cdot+\tau)$ to be uniformly approximated by X, but by $U(\tau)X$. Clearly stationary processes are APU as $S(\varepsilon, X, U)=\mathbb{R}$ for every ε. We note that APU processes might appropriately be called uniformly almost periodically stationary, but almost periodically unitary more clearly reflects the dependence on the collection of unitary operators.

Before continuing with examples we pause to note that it may be possible that $U(\tau)X(t)\notin H(X)$ for some τ and t. Thus the smallest subspace of $L_2(\Omega)$ for which (1.9) makes sense is

$$H(X, U)=\overline{\mathrm{sp}}\{U(\tau)X(t), \tau\in\mathbb{R}, t\in\mathbb{R}\}.$$

We always have $H(X)\subset H(X, U)$, and for stationary processes, $H(X)=H(X, U)$ if U is the shift group for X. From this point forward, we drop the subscript to the norm $\|\cdot\|$, as we always mean the $L_2(\Omega)$ norm, unless otherwise noted.

2. Examples and basic properties

As we have already noted, a weakly stationary process X is APU. The remaining examples are more interesting.

2.1. Almost periodic processes. We begin with uniformly almost periodic (UAP) *processes.* If the process $X:\mathbb{R}\to L_2(\Omega)$ is UAP (see Slutsky [29] and Kawata [21]) it is also APU. These processes, even when not stationary, are APU as may be seen by taking $U(\tau)\equiv I$, which is clearly strongly continuous. Then $S(\varepsilon, X, U)$ defined by (1.9) is identical to (1.6) and thus is RD.

2.2 Continuous PC processes. We have already noted that every PC process has an imbedded unitary operator satisfying (1.8). A strongly continuous one parameter unitary group $\{\tilde{U}(\tau),\ \tau\in\mathbb{R}\}$ may be constructed via the spectral theorem [2]:

$$\tilde{U}(\tau)=\int_{-\pi}^{\pi}\exp(\mathrm{i}\lambda\tau)\,\mathrm{d}E(\lambda) \tag{2.1}$$

where $\tilde{U}(n)=U(n)$ and $\{E(\lambda),\ -\pi<\lambda\leqslant\pi\}$ is the resolution of the identity for U. It is clear that $\|X(t+\tau)-\tilde{U}(\tau/T)X(t)\|=0$ for all t whenever $\tau=nT$, $n\in\mathbb{Z}$ and thus $S(\varepsilon, X, \tilde{U})$ is RD.

2.3. Amplitude scale modulation. If X is wide sense stationary and continuous, and $f:\mathbb{R}\to\mathbb{C}$ is UAP, then $Y(t)=f(t)X(t)$ is APU. Take $U(\tau)$ to be the shift group for the process X. Then

$$\begin{aligned}\|f(t+\tau)X(t+\tau)-U(\tau)[f(t)X(t)]\| &= \|f(t+\tau)X(t+\tau)-f(t)U(\tau)X(t)\|\\ &= |f(t+\tau)-f(t)|\,\|X(t+\tau)\|<\varepsilon\end{aligned}$$

if $|f(t+\tau)-f(t)|<\varepsilon/\sigma_x$ where $\sigma_x=\|X(t)\|$; this shows $S(\varepsilon, Y, U)=S(\varepsilon/\sigma_x, f)$ is RD because the scalar function f is UAP.

2.4. Time scale modulation. If X is wide sense stationary and continuous, and $f:\mathbb{R}\to\mathbb{R}$ is UAP, then $Y(t)=X(t+f(t))$ is APU. Again, take $U(\tau)$ to be the shift group for X. From the stationarity of X, setting $R_x(u)=E\{X(t+u)\overline{X(t)}\}$, we conclude

$$\begin{aligned}\|Y(t+\tau)-U(\tau)Y(t)\| &= \|X(t+\tau+f(t+\tau))-X(t+\tau+f(t))\|\\ &= [2R_x(0)-2r_eR_x(f(t+\tau)-f(t))]^{1/2}.\end{aligned} \tag{2.2}$$

Given an $\varepsilon>0$, then from the continuity of $R_x(u)$ there exists an ε' for which $|f(t+\tau)-f(t)|<\varepsilon'$ implies the quantity in (2.2) does not exceed ε. So $\tau\in S(\varepsilon',f)$ (hence $\sup_{t\in\mathbb{R}}|f(t+\tau)-f(t)|<\varepsilon'$) implies $\tau\in S(\varepsilon, Y, U)$, or $S(\varepsilon',f)\subset S(\varepsilon, Y, U)$. But $S(\varepsilon',f)$ is RD and hence also is $S(\varepsilon, Y, U)$.

These two examples give some indication of the complexity of APU processes. They are neither stationary nor UAP processes. For specific examples of periodic signal modulation that produces cyclostationary (synonymous with PC) processes, see Gardner [11].

2.5. $X(t)=U(t)P(t)$ where $P(t)$ is UAP. Suppose $X(t)=U(t)P(t)$ where $P:\mathbb{R}\to L_2(\Omega)$ is UAP and $\{U(\tau),\ \tau\in\mathbb{R}\}$ is a strongly continuous unitary group. Then

$$\begin{aligned}\|X(t+\tau)-U(\tau)X(t)\| &= \|U(t+\tau)P(t+\tau)-U(\tau)U(t)P(t)\|\\ &= \|P(t+\tau)-P(t)\|\end{aligned}$$

so that $S(\varepsilon, X, U)=S(\varepsilon, P)$ is RD because $P(t)$ is almost periodic.

We are grateful to Professor Karl Peterson for observing the converse; every APU process is given in this manner.

Proposition 1. *A continuous process X is APU with shift group U if and only if*

$$X(t)=U(t)P(t) \tag{2.3}$$

for some almost periodic function P taking values in L_2.

Proof. In view of the preceding remark, we only need to show that every APU process may be given by (2.3). If X is APU with unitary group U, then $Q(t)=U(-t)X(t)$ is easily seen to be continuous and

$$\begin{aligned}\|Q(t+\tau)-Q(t)\| &= \|U(-t-\tau)X(t+\tau)-U(-t)X(t)\|\\ &= \|X(t+\tau)-U(\tau)X(t)\|.\end{aligned}$$

Therefore, $S(\varepsilon, Q)=S(\varepsilon, X, U)$ is RD and hence $Q(t)$ is almost periodic in the sense of Bohr. □

The characterization clearly illuminates the departure from stationarity, in which case we may take P to be the trivial UAP function $P(t)=X(0)$ and write $X(t)=U(t)X(0)$. It is important to note that the representation (2.3) is not unique. For if $X(t)=U(t)P(t)$, we can define a new unitary group by $\tilde{U}(t)=\exp(\mathrm{i}\lambda t)U(t)$ and a new UAP function $\tilde{P}(t)=\exp(-\mathrm{i}\lambda t)P(t)$; clearly $X(t)=\tilde{U}(t)\tilde{P}(t)$.

The following properties may be concluded by the use of Proposition 1 together with known facts about UAP functions taking values in a Hilbert space. We omit the proofs for the sake of brevity.

Proposition 2.

(1) *If $X(t)$ is APU with group U, then*

- (a) *$\|X(t)\|$ is bounded for $t\in\mathbb{R}$;*
- (b) *$X(t)$ is uniformly continuous on $\mathbb{R}$;*
- (c) *U is equi-continuous with respect to $\{X(t),\ t\in\mathbb{R}\}$;*
- (d) *$Z(t,\tau)=U(\tau)P(t)$ is uniformly continuous with respect to $(\tau, t)\in\mathbb{R}^2$;*

(e) *for every* $\varepsilon>0$ *there is an interval* $I_\varepsilon=(-\delta_\varepsilon,\delta_\varepsilon)$ *with* $I_\varepsilon\subset S(\varepsilon,X,U)$;

(f) *for every* $\varepsilon>0$ *there is an* $L(\varepsilon)$ *and* $\delta(\varepsilon)>0$ *such that for every* α *the intersection* $S(\varepsilon,X,U)\cap[\alpha,\alpha+L(\varepsilon)]$ *contains an interval of length* $\delta(\varepsilon)$.

(2) *If* $X:\mathbb{R}\to L_2(\Omega)$ *and* $\{U(\tau),\tau\in\mathbb{R}\}$ *is a shift group of unitary operators, then*

(a) $\varepsilon_2>\varepsilon_1$ *implies* $S(\varepsilon_2,X,U)\supset S(\varepsilon_1,X,U)$;

(b) $\tau\in S(\varepsilon,X,U)$ *iff* $-\tau\in S(\varepsilon,X,U)$;

(c) $\tau_1\in S(\varepsilon_1,X,U)$ *and* $\tau_2\in S(\varepsilon_2,X,U)$, *implies* $\tau_1\pm\tau_2\in S(\varepsilon_1+\varepsilon_2,X,U)$.

(3) *If X and Y are two APU processes with shift groups U and V, then*

(a) *for every* $\varepsilon_1,\varepsilon_2>0$, $S(\varepsilon_1,X,U)\cap S(\varepsilon_2,Y,V)$ *is RD*;

(b) *X and Y orthogonal implies* $Z(t)=X(t)+Y(t)$ *is APU.*

(4) *If* $\{X_n, n=0,1,\ldots\}$ *is a sequence of APU processes each having the common shift group U, and if* $\|X_n(t)-X(t)\|\to 0$ *uniformly with respect to t, then X is APU with shift group U.*

(5) *If* $X(t)$ *is APU with shift group U, then*

(a) $X'(t)$ *uniformly continuous implies* $X'(t)$ *is APU with shift group U*;

(b) *for every real* $a>0$ *the Riemann integral*

$$Y_a(t)=\int_t^{t+a}X(s)\,\mathrm{d}s$$

is APU with shift group U. □

If we denote $M_X=\sup_{t\in\mathbb{R}}\|X(t)\|$, then Proposition 2(1)(a) implies that $R(X)$ (and $\bar{R}(X)$) is bounded as a metric space because

$$\mathrm{d}(X(t_1),X(t_2))=\|X(t_1)-X(t_2)\|_{L_2}\le\|X(t_1)\|_{L_2}+\|X(t_2)\|_{L_2}\le 2M_X$$

for any $t_1,t_2\in\mathbb{R}$. But it is *not* true that $R(X)$ is totally bounded and thus relatively compact as in the case of almost periodic functions (Levitan and Zhikov [24]). Indeed the range of a stationary process may clearly be non-compact as a subset of $L_2(\Omega)$, and such processes are APU.

In the classical proof of Proposition 2(1)(a), the question of boundedness at an arbitrary point t_0 is translated to a closed bounded interval (say $[0,T]$). This notion [4] is also applied to show that UAP functions are uniformly continuous, and is the basis for (1)(b).

A second order process X is called almost periodically correlated in the sense of Gladyshev [13] if $R(u,v)=\langle X(u),X(v)\rangle$ is uniformly continuous in (u,v), and $B(t,\tau)=R(t+\tau,t)$ is a scalar UAP function of t for every τ. If $B(t,\cdot)$, is a UAP function of $t\in\mathbb{R}$ taking values in the Banach space of bounded continuous functions, then X is called UAPC [8]. It is clear that UAPC processes are APC in the sense of Gladyshev.

Proposition 3. *Every APU process X is UAPC.*

Proof. First we may see that boundedness of $B(t,\tau)$ and its continuity with respect to τ for every t follows from the boundedness of $\|X(t)\|$ and the uniform continuity of $X(t)$. To complete the result, we must show that

$$S(B, \varepsilon) = \left\{ \alpha : \sup_{t \in \mathbb{R}} \sup_{\tau \in \mathbb{R}} |B(t+\alpha, \tau) - B(t, \tau)| < \varepsilon \right\}$$

is relatively dense. For fixed t, write

$$X(t+\alpha+\tau) = U(\alpha)X(t+\tau) + \varepsilon(t+\tau, \alpha) \tag{2.4}$$

where $U(\cdot)$ is a shift group for X, and $\{\alpha : \sup_{t \in \mathbb{R}} \|\varepsilon(t, \alpha)\| < \varepsilon\}$ is RD. Then using (2.4) produces

$$\begin{aligned} R(t+\alpha+\tau, t+\alpha) - R(t+\tau, t) = & \langle U(\alpha)X(t+\tau), \varepsilon(t, \alpha)\rangle \\ & + \langle U(\alpha)X(t), \varepsilon(t+\tau, \alpha)\rangle \\ & + \langle \varepsilon(t+\tau, \alpha), \varepsilon(t, \alpha)\rangle. \end{aligned}$$

Application of the Schwarz inequality to each term on the right gives the inequality

$$\begin{aligned} |B(t+\alpha, \tau) - B(t, \tau)| \leq & \, \|U(\alpha)X(t+\tau)\| \, \|\varepsilon(t, \alpha)\| \\ & + \|U(\alpha)X(t+\tau)\| \, \|\varepsilon(t+\tau, \alpha)\| \\ & + \|\varepsilon(t+\tau, \alpha)\| \, \|\varepsilon(t, \alpha)\|. \end{aligned}$$

The unitarity of $U(\cdot)$ and boundedness of $\|X(t)\|$ gives $\sup_{t,\tau} \|U(\alpha)X(t+\tau)\| = M_X$. The definition of $\varepsilon(t+\tau, \alpha)$ gives $\sup_{t,\tau} \|\varepsilon(t+\tau, \alpha)\| < \varepsilon_0$ if $\alpha \in S(\varepsilon_0, X, U)$. Hence

$$\sup_{t \in \mathbb{R}} \sup_{\tau \in \mathbb{R}} |B(t+\alpha, \tau) - B(t, \tau)| \leq 2M_X \varepsilon_0 + \varepsilon_0^2 \equiv K(\varepsilon_0)$$

provided $\alpha \in S(\varepsilon_0, X, U)$. Since $K(\varepsilon_0)$ is monotone for $\varepsilon_0 \geq 0$, we have

$$S(B, \varepsilon) \supset S(K^{-1}(\varepsilon), X). \qquad \square$$

3. Spectral theory and representations

This section presents three representations of APU processes based on $X(t) = U(t)P(t)$. The first is based on the application of Stone's theorem concerning representation of unitary groups [30] to $U(t)$. In our case we obtain

$$X(t) = U(t)P(t) = \int_{-\infty}^{\infty} \exp(\mathrm{i}\gamma t) \, \mathrm{d}E_\gamma(P(t)). \tag{3.1}$$

The integral representation (3.1) may be interpreted as follows; for each fixed t the unitary operator $U(t)$ has the usual representation as an integral with respect to a projection valued measure, and this integral decomposition of $U(t)$ is applied to the vector $P(t)$.

Proposition 4. *A process X is APU if and only if it has the representation*

$$X(t)=\int_{-\infty}^{\infty}\exp(\mathrm{i}\gamma t)Z(\mathrm{d}\gamma, t) \tag{3.2}$$

where for each γ the time dependent vector valued measure $Z(\cdot, t)$ is almost periodic (UAP) in the variable t, and $Z(\cdot, t)$ is orthogonally scattered in that $\langle Z(A, t), Z(B, s)\rangle = 0$ for any s, t whenever $A\cap B=\emptyset$.

Proof. If X is APU, a measure $Z(\cdot, t)$ may be defined by the action of the projections E_γ on the L_2 vector $P(t)$; since any Borel set A defines a projection E_A on a subspace indexed by A we define

$$Z(A, t)=E_AP(t) \tag{3.3}$$

and so for each fixed t, the measure $Z(\cdot, t)$ is orthogonally scattered. Further, since E_A and E_B project onto orthogonal subspaces when $A\cap B=\emptyset$, then for any s, t,

$$\langle Z(A, t), Z(B, s)\rangle = \langle E_AP(t), E_BP(s)\rangle = 0.$$

To see that $Z(A, t)$ is a vector valued almost periodic function (UAP) for each Borel set A, it is enough to observe

$$\|Z(A, t+\tau)-Z(A, t)\| = \|E_AP(t+\tau)-E_AP(t)\| \leq \|P(t+\tau)-P(t)\|$$

and recall that $P(t)$ is UAP.

Suppose now the converse is true. We first observe that $P(t)\equiv Z(\mathbb{R}, t)$ is UAP and define $H(A)=\mathrm{cl}\{\sum_j \alpha_j Z(A_j', t_j), t_j\in\mathbb{R}, A_j'\subset A\}$. If $A\cap B=\emptyset$, then $H(A)\perp H(B)$ from the hypothesis. From this we can define an increasing family of projections operating in the space $H(\mathbb{R})$ by $E_\gamma x=$ Projection of x onto $H((-\infty, \gamma])$. Thus we can write $Z((a, b], t)=E_{(a,b]}P(t)$ for interval $(a, b]$ and

$$X(t)=\int_{-\infty}^{\infty}\exp(\mathrm{i}\gamma t)Z(\mathrm{d}\gamma, t)=\int_{-\infty}^{\infty}\exp(\mathrm{i}\gamma t)\,\mathrm{d}E_\gamma(P(t))=U(t)P(t) \tag{3.4}$$

where $U(t)$ is the group of operators on $H(\mathbb{R})$ generated by the family of projections $\{E_\gamma, \gamma\in\mathbb{R}\}$. □

The second representation for $X(t)=U(t)P(t)$ depends on the fact that every UAP function taking values in Hilbert space is a uniform limit of trigonometric polynomials. That is, for any UAP function $P(t)$ there is a sequence of trigonometric polynomials

$$P_N(t)=\sum_{j=1}^{K(N)} P_{j,K(N)}\exp(\mathrm{i}\gamma_j t) \tag{3.6}$$

with $\lim_{N\to\infty} P_N(t)=P(t)$ uniformly in t (see Corduneanu [7]). It immediately follows from

$$\|X_N(t)-X(t)\| = \|U(t)P_N(t)-U(t)P(t)\| = \|P_N(t)-P(t)\| \tag{3.7}$$

that $X_N(t)=U(t)P_N(t)\to X(t)$ uniformly in t. This is the basis of the following proposition.

Proposition 5. *A process X is APU if and only if it is the uniform limit of processes of the form*

$$X_N(t)=\sum_{j=1}^{K(N)} \eta_{j,K(N)}(t)\exp(\mathrm{i}\gamma_j t) \tag{3.8}$$

where the convergence is uniform with respect to t and $\{\eta_{j,K(N)}(t),\ j=1,\dots,K(N),\ N\in\mathbb{N}\}$ is a collection of jointly wide sense stationary processes and $\{\gamma_j,\ j\in\mathbb{Z}\}$ is a countable set of real numbers.

Proof. Using the preceding remarks, the necessity follows from the definition $\eta_{j,K(N)}(t)=U(t)P_{j,K(N)}$. For the sufficiency, we observe that there must be a group of unitary operators for which $\eta_{j,K(N)}(t)=U(t)\eta_{j,K(N)}(0)$, $j\in\mathbb{Z}$ [28]. Therefore $X_N(t)$ is APU with shift group U and since $X_N(t)\to X(t)$ uniformly, $X(t)$ is also APU with shift group U by Proposition 2.4. □

It is not difficult to show that $\lim_{N\to\infty}\eta_{j,K(N)}(t)$ exists and if we denote this limit (in L_2) as $\eta_j(t)$ then we may consider X to have a Fourier series whose coefficients are stochastic processes

$$X(t)\sim\sum_{j\in\mathbb{Z}}\eta_j(t)\exp(\mathrm{i}\gamma_j t) \tag{3.9}$$

where the frequencies $\Gamma=\{\gamma_j\}$ and coefficients $\{\eta_j(t)\}$ are uniquely determined once $\{U(\tau),\ \tau\in\mathbb{R}\}$ and $P(t)$ are determined. We note that Ogura's representation [27] for harmonizable periodically correlated processes is of the form (3.9) where the $\eta_j(t)$ are bandlimited jointly stationary processes and the convergence is uniform in t; the method used in the proof [14, 16], however relies on methods that have no apparent extension to the general APC case. The final representation is based on the decomposition of $P(t)$ with respect to an orthonormal basis of $H(X)$.

Proposition 6. *A process X is APU if and only if*

$$X(t)=\sum_{j=-\infty}^{\infty}a_j(t)f_j(t) \tag{3.10}$$

where the sequence $\{a_j(t),\ j\in\mathbb{Z}\}$ consists of jointly stationary processes with $E\{|a_j(0)|^2\}=1$ $\forall j$ that are mutually instantaneously orthogonal, $E\{a_j(t)\overline{a_k(t)}\}=0$ for every t and $j\neq k$. The sequence $\{f_j(t),\ j\in\mathbb{Z}\}$ consists of scalar valued UAP functions that satisfy $\sum_j|f_j(t)|^2\leqslant M$ and $\sum_{j>N}|f_j(t)|^2\to 0$ uniformly in t as $N\to\infty$.

Proof. If $X(t)$ is APU, then from the continuity of $X(t)$ there is a countable basis for $H(X)$, and from $P_X(t)=U_X(-t)X(t)$, it is evident that $P_X(t)\in H(X)$. Suppose the vectors $\{\xi_j,\ j\in\mathbb{Z}\}$ are a CONS in $H(X)$. Then we can write

$$P(t)=\sum_j f_j(t)\xi_j \tag{3.11}$$

where for each j, $f_j(t)=\langle P(t), \xi_j\rangle_{L_2(\Omega)}$ is a scalar UAP function and $\|P(t)\|^2=\sum_j|f_j(t)|^2\leq M$. The uniform convergence of $\sum_{j>N}|f_j(t)|^2$ follows from known facts about Hilbert-space-valued almost periodic functions [24, p. 75]. Using $X(t)=U(t)P(t)$ we identify $a_j(t)=U(t)\xi_j$, and note that

$$\langle a_j(t), a_k(t)\rangle=\langle U(t)\xi_j, U(t)\xi_k\rangle=\langle \xi_j, \xi_k\rangle=\delta_{jk}. \tag{3.12}$$

For the converse we know there is a unitary group $\{U(\tau), \tau\in\mathbb{R}\}$ which, for every j, gives $a_j(t)=U(t)\xi_j$ where we have set $\xi_j=a_j(0)$. The uniform convergence of $\sum_{j>N}|f_j(t)|^2$ to 0 ensures that the function $P(t)=\sum_j f_j(t)\xi_j$ is a UAP function taking values in $H(X)$ and evidently $X(t)=\sum_j f_j(t)a_j(t)=U(t)[P(t)]$ as required. □

4. Results related to the correlations

We first shall review the correlation theory for APC processes [17] and then examine the consequence of the inlusion APU ⊂ APC. First, for any correlation of an APC process, the coefficients

$$a(\lambda, \tau)=M\{R(t+\tau, t)\exp(-\mathrm{i}\lambda t)\}=\lim_{A\to\infty}\frac{1}{2A}\int_{-A}^{A}R(t+\tau, t)\exp(-\mathrm{i}\lambda t)\,\mathrm{d}t \tag{4.1}$$

exist for every pair (λ, τ) [13]. Further $\Lambda_\tau=\{\lambda: a(\lambda, \tau)\neq 0\}$ is countable, and for each τ we may write

$$B(t, \tau)=R(t+\tau, t)\sim\sum_{\lambda\in\Lambda_\tau}a(\lambda, \tau)\exp(\mathrm{i}\lambda t), \tag{4.2}$$

where '$\sim$' means that for every τ, $B(t, \tau)$ and the right side of (4.2) have the same frequencies $\lambda_j\in\Lambda_\tau$ and coefficients $a(\lambda_j, \tau)$ determined by (4.1). It is *not* generally true that the partial sums of (4.2) converge uniformly to $B(t, \tau)$ but a sequence of trigonometric polynomials having frequencies in Λ_τ and converging uniformly to $B(t, \tau)$ may be constructed [7, p. 41].

Returning to the coefficients $a(\lambda, \tau)$, the non-negative definite function $a(0, \tau)$ may also be seen from (4.1) to be continuous and therefore Proposition 3 of [17] implies

$$a(\lambda, \tau)=\int_{-\infty}^{\infty}\exp(\mathrm{i}\gamma\tau)r_\lambda(\mathrm{d}\gamma),\qquad \int_{-\infty}^{\infty}|r_\lambda(\mathrm{d}\gamma)|\leq a(0,0)=\int_{-\infty}^{\infty}r_0(\mathrm{d}\gamma). \tag{4.3}$$

That is, each function $a(\lambda, \cdot)$ $\lambda\in\mathbb{R}$ is a Fourier transform of a measure r_λ whose total variation is dominated by that of the finite positive measure r_0. Finally, the continuity of $a(0, \tau)$ together with Proposition 4 of [17] imply that $\Lambda=\bigcup_{\tau\in\mathbb{R}}\Lambda_\tau$ is countable. Thus (4.2) may be rewritten as

$$B(t, \tau)\sim\sum_{j\in\mathbb{Z}}a(\lambda_j, \tau)\exp(\mathrm{i}\lambda_j t). \tag{4.4}$$

The entire family of UAP functions $\{B(\cdot, \tau), \tau\in\mathbb{R}\}$ may be represented, in the sense described above, by a countable set of frequencies $\{\lambda_j\}$ and coefficient functions $\{a(\lambda_j, \tau)\}$.

We shall now determine the connection between the results of Section 3 and the correlation theory. First, from (3.2) and (3.3), for Borel set B we set $F_{s,t}(B)=\langle Z(B,s), Z(B,t)\rangle$ then

$$R(t+\tau,t)=E\{X(t+\tau)\overline{X(t)}\}=\int_{-\infty}^{\infty}\exp(\mathrm{i}\gamma\tau)F_{t+\tau,t}(\mathrm{d}\gamma) \tag{4.5}$$

and it follows from the almost periodicity of $Z(\cdot,t)$ that $F_{t+\tau,t}(B)$ is UAP with respect to the variable t for each τ and B. It may be shown that for every τ and B, the frequencies of the UAP function $F_{t+\tau,t}(B)$ are in the set Λ. Thus for every λ, τ and B the integral

$$r_\lambda(\tau,B)=\lim_{A\to\infty}\frac{1}{2A}\int_{-A}^{A}F_{t+\tau,t}(B)\exp(-\mathrm{i}\lambda t)\,\mathrm{d}t \tag{4.6}$$

exists and $\{\lambda: r_\lambda(\tau,B)\neq 0$ for some $\tau, B\}\subseteq\Lambda$. Define $\Delta(\lambda)=\{(j,k): \lambda=\gamma_j-\gamma_k, (\gamma_j,\gamma_k)\in\Gamma\times\Gamma\}$ where $\Gamma=\{\gamma_j\}$ are the frequencies of the $P(t)$ used in (3.3). The uniform convergence of $P_N(t)$ given by (3.6) to $P(t)$ implies that for fixed τ and B,

$$\lim_{N\to\infty}\langle E_BP_N(t+\tau), P_N(t)\rangle=\langle E_BP(t+\tau), P(t)\rangle=F_{t+\tau,t}(B)$$

uniformly in t. This and the fact $\sum_{j=1}^{\infty}\|P_j\|^2=M\{\|P(t)\|^2\}<\infty$ leads to the conclusion

$$r_\lambda(\tau,B)=\sum_{j,k\in\Delta(\lambda)}\exp(\mathrm{i}\gamma_j\tau)\langle E_BP_j, P_k\rangle. \tag{4.7}$$

The inner products $s_{jk}(B)=\langle E_BP_j, P_k\rangle$ are complex valued measures that are summable on any $\Delta(\lambda)$ in the sense $\sum_p|\langle E_{\mathbb{R}}P_{j_p}, P_{k_p}\rangle|\leqslant\sum_p\|P_{j_p}\|\,\|P_{k_p}\|\leqslant\sum_{j=1}^{\infty}\|P_j\|^2<\infty$, where we have denoted the elements of $\Delta(\lambda)$ by (j_p,k_p), $p=1,2,\ldots,$ and the inequalities follow from the Schwarz inequality applied to L_2 random variables and then to ℓ_2 sequences. These facts make it possible to identify the measure $r_\lambda(B)$ appearing in (4.3) as $r_\lambda(B)=\sum_p s_{j_pk_p}(B+\gamma_{j_p})$ for $(j_p,k_p)\in\Delta(\lambda)$, and where the set $B+\gamma_0=\{\gamma=x+\gamma_0, x\in B\}$.

Continuing now to the implications of Proposition 5, if we define $R_{jk}(u-v)=E\{\eta_j(u)\overline{\eta_k(v)}\}$, then in some sense we expect that

$$R(u,v)=E\{X(u)\overline{X(v)}\}=\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}R_{jk}(u-v)\exp(\mathrm{i}\gamma_ju-\mathrm{i}\gamma_kv). \tag{4.8}$$

To be precise, since $X_N(t)$ given by (3.8) converges uniformly in t to $X(t)$, and $X(t)$ is norm bounded, then for every (u,v),

$$\sigma_N(u,v)=\sum_{j=1}^{K(N)}\sum_{k=1}^{K(N)}R_{jk,K(N)}(u-v)\exp(\mathrm{i}\gamma_ju-\mathrm{i}\gamma_kv) \tag{4.9}$$

converges uniformly to $R(u,v)$ as $N\to\infty$ where

$$R_{jk,K(N)}(u-v)=E\{\eta_{j,K(N)}(u)\overline{\eta_{k,K(N)}(v)}\} \tag{4.10}$$

also converges uniformly to $R_{jk}(u-v)$ as $N \to \infty$. The uniform convergence of $\sigma_N(u, v)$ to $R(u, v)$ means that $\sigma_N(u+\alpha, v+\alpha)$, which is clearly UAP in the variable, α, converges uniformly to $R(u+\alpha, v+\alpha)$ and therefore it follows that for every λ, τ,

$$a_N(\lambda, \tau) = \lim_{A \to \infty} \frac{1}{2A} \int_{-A}^{A} \sigma_N(t+\tau, t) \exp(-\mathrm{i}\lambda t)\, \mathrm{d}t \tag{4.11}$$

converges to $a(\lambda, \tau)$. But $a_N(\lambda, \tau)$ can be evaluated by using (4.9) to give

$$a_N(\lambda, \tau) = \sum_{j,k \in \Delta_N(\lambda)} R_{jk,K(N)}(\tau) \tag{4.12}$$

where $\Delta_N(\lambda) = \{(j, k): \lambda = \gamma_j - \gamma_k,\ (\gamma_j, \gamma_k) \in \Gamma_N \times \Gamma_N\}$. Therefore, in the limit we obtain

$$\lim_{N \to \infty} a_N(\lambda, \tau) = a(\lambda, \tau) = \sum_{j,k \in \Delta(\lambda)} R_{jk}(\tau) \tag{4.13}$$

where $\Delta(\lambda) = \{(j, k): \lambda = \gamma_j - \gamma_k,\ (\gamma_j, \gamma_k) \in \Gamma \times \Gamma\}$. In arriving at this conclusion we use the fact that the set of frequencies $\Gamma_N \subset \Gamma$ must converge to Γ and so $\Delta_N(\lambda) \subset \Delta(\lambda)$ must converge to $\Delta(\lambda)$ as $N \to \infty$; and we also use the fact that the sum appearing in (4.13) may be taken in the absolute sense, from an argument similar to that following (4.7).

To determine the consequences of Proposition 6, we observe that the convergence of (3.10) is uniform in t. This observation may be used to prove the correlation function of an APU process $X(t)$ is of the form

$$R(s, t) = \sum_j \sum_k f_j(s) \overline{f_k(t)} \rho_{jk}(s-t) \tag{4.14}$$

where $\{f_j(t)\}$ are UAP functions with $\sum_j |f_j(t)|^2 \leq M$ for all t and $\{\rho_{jk}(t)\}$ are continuous stationary cross-correlation functions with $\rho_{jj}(0) = 1$, $j \in \mathbb{Z}$, and the convergence is uniform.

If X is *harmonizable* in the sense of Loéve [25] so that $X(t) = \int_{-\infty}^{\infty} \exp(\mathrm{i}\gamma t)\xi(\mathrm{d}\gamma)$ and $R(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(\mathrm{i}\gamma_1 u - \mathrm{i}\gamma_2 v) r_\xi(\mathrm{d}\gamma_1, \mathrm{d}\gamma_2)$ the spectral measure r_ξ essentially describes the non-orthogonality of ξ and gives information about the type of nonstationarity exhibited by X. For example, a harmonizable process $X(t)$ is almost PC in the sense of Gladyshev if and only if r_ξ is concentrated on the set $S_\Lambda = \bigcup_k S_{\lambda_k}$, where $S_{\lambda_k} = \{(\gamma_1, \gamma_2): \gamma_2 = \gamma_1 - \lambda_k\}$ and $\Lambda = \bigcup_{\tau \in \mathbb{R}} \Lambda_\tau$ is the countable set of frequencies required to represent the correlation in (4.4). Further, the restriction of the measure r_ξ to S_{λ_k} may be identified with r_{λ_k} appearing in (4.3). We can see from these remarks that the APU process given by $X_N(t) = U(t)P_N(t)$ for $P_N(t)$ a trigonometric polynomial is strongly harmonizable because we can represent the correlation $E\{X_N(t+\tau)\overline{X_N(t)}\} = \sum_{j=1}^{M(N)} a(\lambda_j, \tau) \exp(\mathrm{i}\lambda_j t)$ as a Fourier integral with respect to $r_{\xi,N}$ that is concentrated on a finite number of diagonal lines. The finiteness of the measures r_{λ_k} forces $r_{\xi,N}$ to be finite. We may thus conclude from (3.6) and Proposition 5 that APU processes are uniform limits of strongly harmonizable processes.

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